Table 1

		l_i/l_1				δ/δ°	m	G°/G
$q = \text{const}$	$h = \text{const}$	1	0.666	1	—	0.7	2	1.21
	$h \neq \text{const}$	1	0.89	0.89	1	0.61	3	1.4
$q = q^\circ e^{-\alpha x}$	$h = \text{const}$	1	0.81	—	—	0.79	1	1.19
	$h \neq \text{const}$	1	0.89	1.3	—	0.7	2	1.37

REFERENCES

1. Andreev, L. V., Mossakovskii, V. I. and Obodan, N. I., On optimal thickness of a cylindrical shell loaded by external pressure. PMM Vol. 36, № 4, 1972.
2. Segal, A. I., Certain results of solving cyclic problems. In coll.: Designing of Spatial Constructions. Moscow, Gosstroiizdat, № 3, 1955.
3. Pontriagin, L. S., The Mathematical Theory of Optimal Processes. (English translation), Pergamon Press, Book № 10176, 1964.
4. Mossakovskii, V. I., Andreev, L. V. and Ziuzin, V. A., Some problems of stability of a cylindrical shell acted upon by nonuniform pressure. Abstracts of papers presented at the 5-th All Union Conference on the Theory of plates and shells. Moscow, 1965.
5. Andreev, L. V. and Obodan, N. I., Stability of cylindrical shells of variable thickness. Prikl. mekhan., Vol. 4, № 5, 1968.
6. Manevich, L. I., Optimal design of a reinforced cylindrical shell under a uniform external load. Dokl. Akad. Nauk USSR, № 7, 1963.

Translated by L. K.

UDC 539.3: 534.231.1

GENERALIZED SOLUTIONS OF BOUNDARY VALUE PROBLEMS IN THE THEORY OF ELASTICITY FOR RANDOM LOADING

PMM Vol. 40, № 3, 1976, pp. 573-576

V. M. GONCHARENKO
(Kiev)

(Received October 24, 1974)

In the present paper we analyze the fundamental static and dynamic boundary value problems of the theory of elasticity, for the case of random loads. We introduce and study various generalized solutions of these problems. The solutions either appear as generalized random functions (random distributions), or belong to the spaces of summable random functions analogous to the Sobolev spaces. These spaces were introduced in [1], and we make use of the imbedding theorem for the random functions proved in that paper to establish the conditions under which the classical solution exists.

1. Certain classes of random functions and distributions. Let Ω be a region in R^n and H be a space of complex random quantities with a finite second moment and Hilbertian with respect to the scalar product $(\xi, \eta) = M\xi\bar{\eta}$. The random second order functions defined in Ω can be regarded as functions with values in H. The spaces $C^m(\Omega, H)$ and $C_0{}^m(\Omega, H)$ consist of H-valued functions with derivatives in Ω of up to the m-th order, strongly defined (with respect to the norm of H, i.e. in the RMS sense) and continuous in the same sense. The spaces are analogous to the spaces $C^m(\Omega)$ and $C_0{}^m(\Omega)$. The space $L_p(\Omega, H)$ $(p > 1)$ is formed by the random functions $u(x)$ which are strongly measurable with respect to the norm of H and such, that the function $x \to \| u(x) \|_H$ belongs to $L_p(\Omega)$.

If the region Ω is bounded, then the functions belonging to $L_p(\Omega, H)$ are Bochner-integrable in Ω. The mapping

$$\varphi \to \int\limits_\Omega u(x)\,\varphi(x)\,dx, \qquad \varphi \in D(\Omega)$$

belongs to $\{D(\Omega), H\}$, i.e. it represents a linear, continuous mapping of the basic Schwartz functions $D(\Omega)$ on H. Such mappings are called the second order random distributions. Operations which can be performed on these distributions are described in [1 — 3]. For functions belonging to $L_p(\Omega, H)$ we can consider derivatives of any order in the sense of $\{D(\Omega), H\}$. In [1] the author introduces the spaces $W_p{}^l(\Omega, H)$ of random functions belonging to $L_p(\Omega, H)$ the derivatives of which also belong, in the same sense, to $L_p(\Omega, H)$ up to the l-th order. The spaces $W_p{}^l(\Omega, H)$ are Banach spaces with respect to the norm

$$\Big[\int\limits_\Omega \sum_{|\alpha|\leqslant l} \| D^\alpha u(x) \|_H^p dx\Big]^{1/p}$$

They are analogs of the Sobolev spaces $W_p{}^l(\Omega)$ and have the same properties. In particular [1], when $l > n/2 + \sigma$, the space $W_2{}^l(\Omega, H)$ is continuously imbedded in $C^{[\sigma]}(\Omega, H)$. The manifold $C_0{}^\infty(\Omega, H)$ is dense in $L_2(\Omega, H)$, but the closure $W_2{}^{0l}(\Omega, H)$ of the manifold $C_0{}^\infty(\Omega, H)$ on the norm of $W_2{}^l(\Omega, H)$ is a characteristic subspace of $W_2{}^l(\Omega, H)$.

The criterion of the random function $u(x)$ belonging to the spaces $L_2(\Omega, H)$ and $W_2{}^l(\Omega, H)$ can be formulated in terms of its covariational function

$$K(x, y) = Mu(x)\,u(y)$$

A random function $u_1(x)$ equivalent to $u(x)$ and belonging to $L_2(\Omega, H)$ will exist if and only if $K(x, x)$ is integrable in Ω. If the generalized derivatives

$$D_x{}^\alpha D_y{}^\alpha K(x, y)\,|_{y=x} \qquad (|\alpha| \leqslant l)$$

are also integrable, then $u_1(x)$ belongs to $W_2{}^l(\Omega, H)$.

Below we shall find it necessary to use the spaces of random vector functions and distributions $\mathbf{u} = (u_1, u_2, \ldots u_n)$ which can be obtained by replacing H by H^n. We shall denote the norms of the spaces $L_2(\Omega, H^n)$ and $W^l{}_2(\Omega, H^n)$ by $\|\cdot\|_0$ and $\|\cdot\|_l$, and the scalar products by $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_l$.

2. Static problems with homogeneous boundary conditions. The boundary value problems of the theory of elasticity are associated with the following differential expressions:

$$Au = \left\{ - \sum_{k, \alpha, \beta=1}^{n} \frac{\partial}{\partial x_k} (c_{kl\alpha\beta} \varepsilon_{\alpha\beta}) \right\}_{l=1}^{n}, \quad \varepsilon_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)$$

Let us consider e.g. the first boundary value problem in a bounded region Ω with a piecewise smooth boundary $\partial\Omega$

$$A\mathbf{u} = \mathbf{f} \quad (x \in \Omega), \ \mathbf{u} = 0 \ (x \in \partial\Omega) \tag{2.1}$$

If $c_{kl\alpha\beta}$ are determinate and $\mathbf{f} = \mathbf{f}(x)$ is a given random function of second order, then the generalized solution of the problem (2.1) can be introduced in analogy with the determinate case using the energy method [4]. Let us consider the operator A_0: $L_2(\Omega, H^n) \to L_2(\Omega, H^n)$ acting according to the formula $A_0 \mathbf{u} = A\mathbf{u}$, with $C_0^\infty(\Omega, H^n)$ serving as its domain of definition. Since $C_0^\infty(\Omega, H^n)$ is dense in $L_2(\Omega, H^n)$ and, as we can easily verify, $(Au, v)_0 = (u, Av)_0$ for all $u, v \in C_0^\infty(\Omega, H^n)$, it follows that the operator A_0 is symmetric and it can be shown that it is positive-definite. Therefore there exists according to Friedrichs a selfconjugate expansion A of operator A_0. The equation $A\mathbf{u} = \mathbf{f}$ is single-valued and has a correct solution for all $\mathbf{f} \in L_2(\Omega, H^n)$. This solution represents a generalized solution of the stochastic problem (2.1).

The energy space of the operator A_0 coincides with $W_2^{01}(\Omega, H^n)$ and the energy norm is equivalent to the norm of the space $W_2^1(\Omega, H^n)$. The energy norm is generated by the scalar product

$$[\mathbf{u}, \mathbf{v}] = (A\mathbf{u}, \mathbf{v})_0 = 2\mathbf{M} \int_\Omega W dx, \quad W = \frac{1}{2} \sum c_{kl\alpha\beta} \varepsilon_{kl} \varepsilon_{\alpha\beta} \tag{2.2}$$

The generalized solution $\mathbf{u}$ causes the functional of mathematical expectation

$$F(\mathbf{v}) = {}^1/_2 (A\mathbf{v}, \mathbf{v})_0 - (\mathbf{v}, \mathbf{f})_0 \tag{2.3}$$

to assume its minimum value. Similarly to the determinate case, the generalized solution of the stochastic boundary value problem (2.1) can be obtained using various projection, variational and variational-difference methods. The properties of the operator A make it possible to obtain, for the stochastic systems, analogous results on convergence in terms of the probability norms $\| \cdot \|_0$, $\| \cdot \|_1$ and $[(A\mathbf{u}, \mathbf{u})]^{1/2}$.

The generalized solution introduced above can also be investigated in the case when $c_{kl\alpha\beta}$ are random functions. The previous results are all retained provided that $c_{kl\alpha\beta}$ belong to the space of random functions measurable and bounded in the measure equal to the product of the probability measure and the Lebesgue measure in Ω. Other types of the boundary conditions can also be investigated.

3. Cauchy problem with random data. Let $\mathbf{u}_0, \mathbf{u}_1 \in \{D(R^n), H^n\}$, $\mathbf{f}(x, t) \in \{D(R^{n+1}), H^n\}$, with $\mathbf{f} = 0$ when $t < 0$. Under the Cauchy problem with random data we understand the problem of determining the random distribution $\mathbf{u} \in \{D(R^{n+1}), H^n\}$ which satisfies the equation

$$\mathbf{u}^{\cdot\cdot} + A\mathbf{u} = \mathbf{f}_1 \equiv \mathbf{f} + \mathbf{u}_0 \times \delta^{\cdot}(t) + \mathbf{u}_1 \times \delta(t) \tag{3.1}$$

and the condition that $\mathbf{u} = 0$ when $t < 0$.

Let $E(x, t)$ be the fundamental matrix of dynamic equations of the theory of elasticity. The solution of the Cauchy problem formulated above is unique, and can be written in the form of convolution

$$\mathbf{u} = E * \mathbf{f}_1 = E * \mathbf{f} + [E * (\mathbf{u}_0 \times \delta)]^{\cdot} + E * (\mathbf{u}_1 \times \delta) \tag{3.2}$$

The proof of existence of the convolution $E * \mathbf{f}_1$ of the generalized function $E \in D'(R^{n+1})$ with the random distribution $\mathbf{f}_1$ can be carried out in the manner similar to the proof in [5] for the wave equation and nonrandom functions.

The formula (3.2) makes possible the determination of the probability characteristics of the solution. Let e.g. $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{f}$ be independent and have characteristic functionals Φ_0, Φ_1 and $\Phi_\mathbf{f}$. Then the characteristic functional of the solution will be given by

$$\Phi_\mathbf{u}(\varphi) = \Phi_\mathbf{f}(\varphi_E)\,\Phi_1[\varphi_E(x, 0)]\,\Phi_0[\varphi_E^{\cdot}(x, 0)],\ \varphi_E = E^{\vee} * \varphi,\ \varphi \in [D(R^{n+1})]^n \quad (3.3)$$

where $^{\vee}$ denotes inversion.

The solution (3.2) represents a random distribution. It will be interesting to find out the conditions under which this distribution is generated by a random function of various classes. We have the following theorems:

Theorem 1. If $\mathbf{f} \in C\{[0, T], W_2^l(\Omega, H^n)\}$, $\mathbf{u}_0 \in W_2^{l+1}(\Omega, H^n)$ and $\mathbf{u}_1 \in W_2^l(\Omega, H^n)$ for every $T > 0$ and the regions $\Omega \subset R^n$, then the solution of the Cauchy problem is a random function $\mathbf{u}(x, t)$ belonging to $W_2^l(\Omega, H^n)$ for all $0 < t < T$. Moreover, $\mathbf{u}^{\cdot} \in W_2^{l-1}(\Omega, H^n)$ and the solution $t \to \{\mathbf{u}, \mathbf{u}^{\cdot}\}$ can be interpreted as being a continuous curve in the space $W_2^l(\Omega, H^n) \times W_2^{l-1}(\Omega, H^n)$.

Theorem 2. If under the conditions of Theorem 1 $l > (n+1)/2 + 2$, then the solution of the Cauchy problem becomes the classical solution, i.e. it represents a random function of the class $C^2(t > 0, H^n) \cap C^1(t \geqslant 0, H^n)$.

The proof of Theorem 1 can be carried out according to the classical scheme [6] which involves regularizing the random distributions and making a priori estimates in terms of the norms $\|\cdot\|_l$. The proof of Theorem 2 follows from the imbedding theorem for random functions. The conditions of Theorem 2 can be weakened by introducing e.g. a space $W_2^l(\Omega, H^n)$ with fractional l.

4. Mixed boundary value problem. Let us consider the boundary value problem

$$\begin{aligned} &\mathbf{u}^{\cdot\cdot} + A\mathbf{u} = \mathbf{f}, \quad (x, t) \in \Omega \times R_+ \qquad (4.1)\\ &\mathbf{u} = 0, \quad (x, t) \in \partial\Omega \times R_+ \\ &\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \mathbf{u}^{\cdot}(x, 0) = \mathbf{u}_1(x) \end{aligned}$$

where $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{f}$ are given random functions. Depending on the conditions imposed on their covariant matrices, we have either the classical solution, or various generalized solutions of the stochastic problem (4.1).

Theorem 3. Let the derivatives $D_x^\alpha D_y^\alpha$ of the covariant functions of the components of $\mathbf{u}_0(x)$ and $\mathbf{u}_1(x)$ be integrable in Ω, when $y = x$, for all $0 \leqslant |\alpha| \leqslant l+1$ and $0 \leqslant |\alpha| \leqslant l$, respectively, and let the analogous derivatives $(0 \leqslant |\alpha| \leqslant l)$ of covariant functions $K_{fs}(x, t; y, \tau)$ of the components of f_s be integrable in Ω with respect to x for $y = x$, $\tau = t$ and all $t > 0$. Let the condition of continuity in the large also hold

$$\lim_{\tau \to t} \int_\Omega \sum_{0 \leqslant |\alpha| \leqslant l} D_x^\alpha D_y^\alpha [K_{fs}(x, t; y, t) + K_{fs}(x, \tau; y, \tau) - 2K_{fs}(x, t; y, \tau)]_{y=x}\, dx = 0$$

Then a unique generalized solution of class $C\{[0, \infty), W_2^l(\Omega, H^n)\}$ exists. If in addition $l \geqslant 5$ $(n = 3)$ or $l \geqslant 4$ $(n = 2)$, then a classical solution exists.

Solution of the stochastic problem (4.1) of class $C\{[0, \infty), W_2^l(\Omega, H^n)\}$ and the proof of Theorem 3 are analogous to those given in [6] for a determinate mixed boundary value problem for the second order hyperbolic equation. Using the methods given in [7], we can obtain less strict conditions of existence of the generalized solution of class $C\{[0, \infty), W_2^l(\Omega, H^n)$.

Theorem 4. If $\mathbf{u}_0 \in W_2^{01}(\Omega, H^n)$, $\mathbf{u}_1 \in L_2(\Omega, H^n)$ and $\mathbf{f} \in L_2[\Omega \times (0, T), H^n]$, then there exists a unique generalized solution of the stochastic problem (4. 1) of class $C\{[0, T], W_2^{01}(\Omega, H^n)\}$. Moreover, $\mathbf{u}^\cdot \in C\{[0, T], L_2(\Omega, H^n)\}$. The dependence of $\{\mathbf{u}, \mathbf{u}^\cdot\}$ on $\{\mathbf{f}, \mathbf{u}_0, \mathbf{u}_1\}$ is continuous just as the mapping of the space $L_2[\Omega \times (0, T), H^n] \times W^{01}_2(\Omega, H^n) \times L_2(\Omega, H^n)$ onto the space $C\{[0, T], W_2^{01}(\Omega, H^n) \times L_2(\Omega, H^n)\}$.

The results obtained remain valid if $c_{kl\alpha\beta}$ are random functions belonging to the space L_∞, and they also hold for a number of the boundary value problems of the theory of plates and shells.

REFERENCES

1. Goncharenko, V. M., Certain classes of vector-valued generalized functions and their application towards the boundary value problems with respect to random functions. Ukr. matem. zh., Vol. 27, № 2, 1975.
2. Ito, K., Stationary random distribution. Mem. Coll. Sci. Univ. Kyoto, Vol. 28, 1953.
3. Gel'fand, I. M. and Vilenkin, N. Ia., Certain Applications of Harmonic Analysis. Fitted Hilbert Spaces. Moscow, Fizmatgiz, 1961.
4. Mikhlin, S. G., The Problem of the Minimum of a Quadratic Functional. (English translation), San Francisco, Holden-Day, 1965.
5. Vladimirov, V. S., Equations of Mathematical Physics, Moscow, "Nauka", 1971.
6. Sobolev, S. L., Applications of Functional Analysis in Mathematical Physics. (English translation) American Math. Society, Vol. № 7, Providence, R. I., 1963.
7. Lions, J. L. and Magenes, E., Problèmes aux Limites non Homogènes et Applications. Paris, Dunod, 1968-70.

Translated by L. K.